Equivariant Minimal Surfaces in Complex Hyperbolic Spaces

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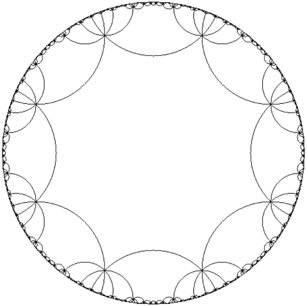
Technical beginnings: Equivariant Minimal Surfaces

Let Σ be a closed, oriented surface of genus $g \geq 2$.



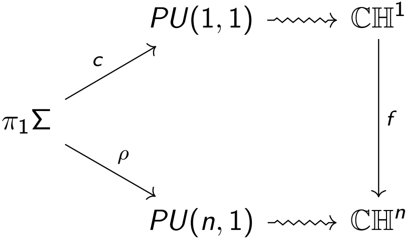
Let $\mathbb{CH}^n$ be n-dimensional complex hyperbolic space considered as $\mathbb{PC}_-^{n,1}$ equipped with

$$\langle v, v \rangle = \sum_{i=1}^{n} v_i \overline{v}_i - v_{n+1} \overline{v}_{n+1}.$$

$PU(n, 1)$ is the group of orientation preserving isometries of $\mathbb{CH}^n$.

Technical beginnings: Equivariant Minimal Surfaces

Let $f : \mathbb{CH}^1 \to \mathbb{CH}^n$ be a minimum immersion,
$c : \pi_1\Sigma \to PU(1,1)$ be a Fuchsian representation,
and $\rho : \pi_1\Sigma \to PU(n,1)$ be an indecomposable representation,
such that f **intertwines** the actions of c and ρ:



Definition

An **equivariant minimal surfaces** in $\mathbb{CH}^n$ is an equivalence class $[f, c, \rho]$

Technical beginnings: Equivariant Minimal Surfaces

Let $\mathcal{M}$ be the moduli space of $\mathbb{CH}^n$ equivariant minimal surfaces:

$$\mathcal{M} = \{[f, c, \rho]\}$$

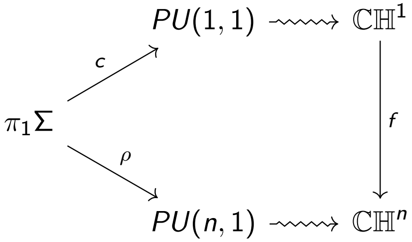
Corlette proved we have an injective map:

$$\begin{aligned} \mathcal{M} &\to \mathcal{T}_g \times \mathcal{R}(\pi_1\Sigma, PU(n,1)) \\ [f, c, \rho] &\mapsto ([c], [\rho]) \end{aligned}$$

where $\mathcal{T}_g$ denotes Teichmüller space and $\mathcal{R}(\pi_1\Sigma, PU(n,1))$ is the character variety

What you need to know: Equivariant Minimal Surfaces

We want to know all the different ways a complex line can be embedded in n-dimensional complex hyperbolic space while minimising its area.



The useful bit from the 80s: Non-Abelian Hodge Theory

Representations	$\longleftrightarrow$	Flat connections	$\longleftrightarrow$	Harmonic metrics	$\longleftrightarrow$	Higgs bundles

The other useful bit from the 80s: NAH+

Representations $\longleftrightarrow$ Flat connections $\longleftrightarrow$ Harmonic metrics $\longleftrightarrow$ Higgs bundles

Harmonic metrics $\updownarrow$ Harmonic Sequences

Another scary technical slide: Harmonic Sequence

Let L be take the tautological line subbundle of the $\mathbb{C}^{n,1}$ trivial bundle over $\mathbb{CH}^n$ with canonical flat connection ∇'.
$f = f_0 : \Sigma \to \mathbb{CH}^n$ pulls L back to a holomorphic line subbundle ℓ_0 of the trivial bundle $\Sigma \times \mathbb{C}^{n,1}$ with a flat hermitian connection ∇.
Define

$$\begin{aligned} A_0 : \ell_0 &\to \ell_0^{\perp} \otimes T^*_{1,0}\Sigma \\ \sigma_0 &\mapsto \pi_0^{\perp} \nabla_Z(\sigma_0), \end{aligned} \tag{1}$$

Similarly, let

$$\begin{aligned} \overline{A}_0 : \ell_0 &\to \ell_0^{\perp} \otimes T^*_{0,1}\Sigma \\ \sigma_0 &\mapsto \pi_0^{\perp} \nabla_{\overline{Z}}(\sigma_0). \end{aligned} \tag{2}$$

Provided $A_0 \neq 0 \neq \overline{A}_0$, they define unique complex line bundles ℓ_1 and ℓ_{-1} corresponding to functions $f_1, f_{-1} : \Sigma \to \mathbb{PC}^{n,1}$.

Everything works: Harmonic Sequence

Lemma

If f_0 is harmonic, then so are f_1 and f_{-1}.

Inductively, we continue to define maps A_i and $\overline{A}_i$ and successively build up harmonic maps $\{f_i : \Sigma \to \mathbb{PC}^{n,1}\}$.

Definition

Let $I = \{i \in \mathbb{Z} | f_i \text{ is well defined}\}$, then we call $\{f_i\}_{i \in I}$ a **harmonic sequence** and $\{\ell_i\}_{i \in I}$ form the corresponding bundle sequence.

Some (not-well) hidden Linear algebra: Harmonic Sequence

Definition

If the image of f_0 does not lie in a totally geodesic copy of $\mathbb{CH}^k$ for some $k < n$, we call f_0 **linearly full**.

If all elements of the bundle sequence are mutually orthogonal and f_0 is linearly full then the sequence is finite and called **superminimal**.

$$0 \longrightarrow \ell_{p-n} \longrightarrow \ldots \longrightarrow \ell_{-1} \longrightarrow \ell_0 \longrightarrow \ell_1 \longrightarrow \ldots \longrightarrow \ell_p \longrightarrow 0$$

Winding back: Non-Abelian Hodge Theory

Representations $\longleftrightarrow$ Flat connections $\longleftrightarrow$ Harmonic metrics $\longleftrightarrow$ Higgs bundles

Harmonic metrics $\updownarrow$ Harmonic Sequences

Some more long definitions: Higgs bundles

Definition

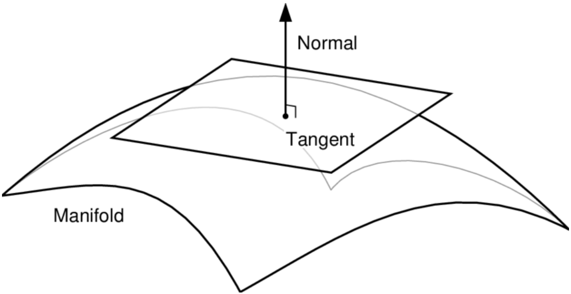
A **G-Higgs bundle** over a closed Riemann surface X is a pair (E, Φ) where $E \to X$ is a holomorphic principal bundle with a $H^{\mathbb{C}}$ structure for H a maximal compact subgroup of G, and $\Phi : E \to E \otimes K$ is a **Higgs field**, where K is the canonical bundle.

Here we have $G = PU(n, 1)$ so (E, Φ) orthogonally split $E = V \oplus L$ where L is a holomorphic line bundle and V a rank n holomorphic bundle while $\Phi = (\phi_1, \phi_2)$ where $\phi_1 \in H^0(X, \mathrm{Hom}(L, V) \otimes K)$ and $\phi_2 \in H^0(X, \mathrm{Hom}(V, L) \otimes K)$

More understandably: Higgs bundle

A **vector bundle** attaches a vector space to each point of the space.

The Higgs bundles we care about here are vector bundles which consist of a line orthogonal to a (nice) n-dimensional space and a map which lets you twist them nicely.



The useful bit of Higgs bundles

There is a $\mathbb{C}^*$ action on $\mathcal{M}_{Higgs}$ given by

$$\begin{aligned} \mathbb{C}^* \times \mathcal{M}_{Higgs} &\to \mathcal{M}_{Higgs} \\ (z, [(E, \Phi)]) &\mapsto [(E, z\Phi)]. \end{aligned}$$

The critical submanifolds of this correspond to **Hodge bundles** which have further splitting:

$$E = V_2 \oplus L \oplus V_1, \qquad \Phi = (\phi_1, \phi_2)$$

and

$$V_2 \otimes K^{-1} \xrightarrow{\phi_2} L \xrightarrow{\phi_1} V_1 \otimes K.$$

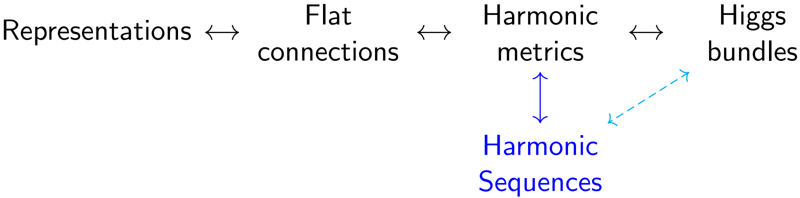
Higgs bundles and Morse theory

Lemma

Every Higgs bundle $(V \oplus 1, \Phi)$ corresponding to $[f, c, \rho]$ can be written as an extension of a Hodge bundle $(V_1 \oplus 1 \oplus V_2, \Phi')$ and lies in that Hodge bundle's unstable manifold.

Instead of considering Higgs bundles, we just need to consider Hodge bundles and an extension $[\alpha] \in H^1(\Sigma, \mathrm{Hom}(V_2, V_1))$.

Winding back: Non-Abelian Hodge Theory



Higgs bundles and Harmonic Sequences

Lemma

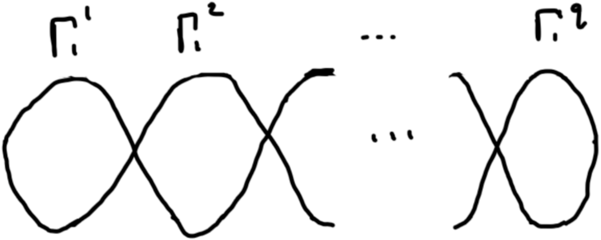
The harmonic sequence corresponding to an equivariant minimal surface $[f, c, \rho]$ is superminimal if and only if the $PU(n,1)$-Higgs bundle (E, Φ) corresponding to $[f, c, \rho]$ is a Hodge bundle.

So $(V_1 \oplus L \oplus V_2, (\phi_1, \phi_2))$ corresponds to

$$0 \longrightarrow \ell_{p-n} \longrightarrow \ldots \longrightarrow \ell_{-1} \xrightarrow{\phi_2} \ell_0 \xrightarrow{\phi_1} \ell_1 \longrightarrow \ldots \longrightarrow \ell_p \longrightarrow 0$$

Classifying Hodge

Superminimal Harmonic Sequences are classified by (Γ_0, Γ_1) and we cam show that Γ_0 is fixed for a given submanifold so:



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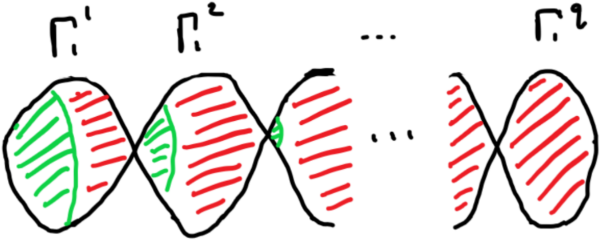

Extensions: Isotropy Order

$$\ldots \longrightarrow \ell_{p-n} \longrightarrow \ldots \longrightarrow \ell_{-1} \xrightarrow{\phi_2} \ell_0 \xrightarrow{\phi_1} \ell_1 \longrightarrow \ldots \longrightarrow \ell_p \longrightarrow \ldots$$

The **isotropy order** of a harmonic sequence is the maximum number of consecutive ℓ_i which are orthogonal.
The **isotropy order** of a Higgs bundle is the isotropy order of its harmonic sequence.

Extensions: Isotropy Order

Higgs bundles of every possible isotropy order exist and we can determine the isotropy order from the extension β:

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$$\phi_{21}\beta_1^*\phi_{11}\sigma_0 = \phi_{22}\phi_{12}\sigma_0$$
$$\phi_{21}\beta_2^*\phi_{12}\sigma_0 = \phi_{22}\partial_1\phi_{11}\sigma_0$$
$$\phi_{22}\beta_3^*\phi_{12}\sigma_0 = \partial_2\phi_{11}\sigma_0$$

Extensions: Isotropy Order

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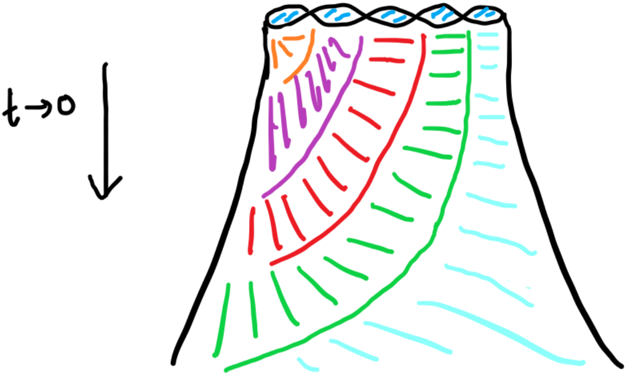

Thank you